ON MAHARAM'S CONDITIONS FOR MEASURE

BY

J. L. HODGES, JR., AND ALFRED HORN

A recent paper of Maharam(1), referred to below as (M), provides a set of necessary and sufficient conditions that a countably complete Boolean algebra have a measure(2). These conditions are (I) a kind of infinite distributive law, and (II, IIIa, IIIb) conditions stating the existence of sets of elements having certain properties. The purpose of this note is to point out that (I) may be omitted; more precisely, we prove that (II) and (IIIa) imply (I).

We shall denote the Boolean algebra as well as the set of its elements by E, the zero element by o, the complement of $A \subset E$ by $Co(A)$, and symmetric difference by $\oplus$. The suffixes k, m, n range over the positive integers. Instead of $\inf_n \{\sup_{m \geqq n}(x_m \oplus x)\} = o$, we shall write $x_n \to x$; and the closure of $A \subset E$ with respect to this convergence will be denoted by $\overline{A}$. A set $X \subset E$ will be said to be of type (C) if o is not a member of $\overline{X}$.

THEOREM. *If E satisfies the following conditions:*

(II) *There exists a countable number of sets $C_1, C_2, \cdots$ of type* (C) *such that each countable set of type* (C) *is contained in some C_k.*

(IIIa) *The sets $\{C_i\}$ in* (II) *can be chosen so as to satisfy $Co(C_{k+1}) \oplus Co(C_{k+1}) \subset Co(C_k)$ for every k;*
then E satisfies condition (I) of (M).

Proof. In the proof of Theorem 2 of (M) it is shown that (I) is a consequence of the condition that the closure of $\overline{A} \subset \overline{A}$ for every $A \subset E$. Let $A \subset E$, $x \in$ the closure of $\overline{A}$, and choose a sequence $\{x_n\}$ of elements of $\overline{A}$ such that $x_n \to x$. For every n, choose a sequence $\{x_{m,n}\}$ of elements of A such that $x_{m,n} \to x_n$. Observe that $x_n \oplus x \to 0$ is equivalent to $x_n \to x$. Let k be a positive integer, and find N such that $(x_N \oplus x) \in Co(C_{k+1})$; such an N must exist since otherwise C_{k+1} would contain a sequence of elements converging to o in contradiction to (II). Similarly find M such that $(x_{M,N} \oplus x_N) \in Co(C_{k+1})$. Observing $(x_{M,N} \oplus x) = (x_{M,N} \oplus x_N) \oplus (x_N \oplus x)$ we conclude by (IIIa) that $(x_{M,N} \oplus x) \in Co(C_k)$. It follows from (II) that o must belong to the closure of the set of elements of the form $(x_{m,n} \oplus x)$ since this set is countable and is not contained in any C_k. We can thus find sequences $\{m_i\}$ and $\{n_i\}$ of positive

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(1) Dorothy Maharam, *An algebraic characterization of measure algebras*, Ann. of Math. vol. 48 (1947) pp. 154–167.

(2) That is, a countably additive and strictly positive measure in the sense of Alfred Horn and Alfred Tarski, *Measures in Boolean algebra*, Amer. Math. Soc. vol. 64 (1948) pp. 467–497. In the latter paper, one of the conditions of (M) is examined from another standpoint.

integers such that $(x_{m_i,n_i} \oplus x) \to 0$. Therefore $x_{m_i,n_i} \to x$ and $x \in \bar{A}$.

The proof of Theorem 2 of (M) requires that E be atomless, but our result remains valid without this assumption. Condition (II) assures that the set of atoms of E is countable, and hence that E is isomorphic to the direct product of an atomless algebra E_1 and an atomistic (that is, one in which every element is a sum of atoms) algebra E_2. The algebra E_1 satisfies conditions (II) and (IIIa) (using the sets C_n of E with the atoms removed), and hence satisfies condition (I). Moreover it is well known that an atomistic algebra satisfies (I) (even without the monotony condition). It is easy to check that condition (I) is preserved under direct product.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF.
UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIF.